THEORY OF WEAK TURBULENCE OF WAVES ON A FLUID SURFACE

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The nonlinear interaction of waves in a fluid of finite depth is discussed. Forbidden decay processes in the gravitational portion of the spectrum are eliminated from the Hamiltonian by means of a canonical transformation. This provides an opportunity to obtain a kinetic equation which takes into account scattering of capillary waves by gravitational waves, in addition to decays in the subsystem of gravitational waves. The distribution $N_k \sim P^{1/2}h^{1/4}k^{-4}$ is obtained for capillary waves in shallow water with constant flow of energy P with respect to the spectrum in the space of the wave numbers k. The interaction of the gravitational and capillary turbulence spectra is discussed. An induced distribution of gravitational waves is found which results from their interaction with capillary waves. It is an increasing function of the wave numbers q in the region bounded by the capillary constant k_0, $N_q \sim q^{9/4}(q < k_0)$. The coupling of spectra in the gravitational and capillary regions and the conversion from slightly turbulent distributions to universal distributions are discussed.

1. The existence of slightly turbulent local distributions of the Kolmogorov type is possible in a system of surface waves (because of velocity dispersion). For deep water, such distributions, which correspond to a constant flow of energy P in the high-frequency region, were found to be [1, 2]

$$N(\mathbf{k}) = P^{1/3} k^{-4}, \quad h^{-1} \ll k \ll k_0 \tag{1.1}$$

$$N(\mathbf{k}) = P^{1/2} k^{-4} V_k^{-1/2} \sim P^{1/2} k^{-17/4}. \quad k_0,\ h^{-1} \ll k \tag{1.2}$$

Here $V_k = \omega_k/k$ is the phase velocity of the waves, $\omega_k = (gk + (\alpha/\rho)k^3)^{1/2}$ is the dispersion law, and $N(\mathbf{k})$ is the density of the number of waves with the wave vector **k**, which determines the energy density in **k**-space $\omega_k N(k)$. In addition to Eq. (1.1), there is a distribution with constant flow Q of number of waves ("particles")* for gravitational waves,

$$N(\mathbf{k}) = Q^{1/3} \omega_k^{1/3} k^{-4} \sim Q^{1/3} k^{-23/6}, \quad h^{-1} \ll k \ll k_0 \tag{1.3}$$

in accordance with the fact that the number of waves in this region is an integral of the motion.

The separation of the gravitational ($k \ll k_0 \equiv \sqrt{\rho g/\alpha}$) and capillary ($k \gg k_0$) regions and the deep-water condition ($kh \gg 1$) are essentially used in the determination of the solutions (1.1)-(1.3), which is associated with the self-similarity of the equations in these regions.

In this paper, the weakly turbulent distribution of capillary waves in shallow water is determined (Sec. 4) and the coupling and interactions of turbulence spectra in various self-similar regions are discussed (Secs. 5, 6). The roughness spectra are found from the kinetic equation for $N(\mathbf{k})$ describing the random ensemble of waves in the theory of weak turbulence [3, 4]. The derivation of the kinetic equation from the equations of motion is discussed in Sec. 3. The equations of motion and the matrix elements for wave interactions in a fluid of finite depth are obtained in Sec. 2 in Hamiltonian variables.

*V. E. Zakharov, Doctoral Dissertation, Novosibirsk (1967).

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2. As has been shown [1, 2, 5], the potential motion of a heavy incompressible fluid with a free surface $z = \zeta(\mathbf{r}, t)$ filling the half-space $z < \zeta$ can be described in Hamiltonian variables which are the elevation of the surface $\zeta(\mathbf{r}, t)$ and the potential of the velocity at the surface $\varphi|_{z=\zeta} = \psi(\mathbf{r}, t)$. These variables also remain Hamiltonian for a fluid of finite depth. The equations

$$\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2\right]_{z=\zeta} + g\zeta = \frac{\alpha}{\rho}\nabla_\perp\left(\frac{\Delta_\perp\zeta}{\sqrt{1+(\nabla_\perp\zeta)^2}}\right) \tag{2.1}$$

$$\partial\zeta/\partial t = (\partial\varphi/\partial z - \nabla_\perp\varphi\nabla_\perp\zeta)_{z=\zeta}, \quad \nabla_\perp \equiv (\partial/\partial x, \ \partial/\partial y) \tag{2.2}$$

through the use of the volume equation

$$\Delta\varphi = 0, \quad -h < z < \zeta, \quad \left.\frac{\partial\varphi}{\partial z}\right|_{z=-h} = 0 \tag{2.3}$$

can be written in the form

$$\dot\zeta = \delta(E/\rho)/\delta\psi, \quad \dot\psi = -\delta(E/\rho)/\delta\zeta \tag{2.4}$$

$$E = \frac{\rho}{2}\int d\mathbf{r}\int_{-h}^{\zeta} dz\,(\nabla\varphi)^2 + \frac{\rho g}{2}\int d\mathbf{r}\zeta^2 + \alpha\int d\mathbf{r}\left(\sqrt{1+(\nabla_\perp\zeta)^2} - 1\right) \tag{2.5}$$

where E is the total energy of the system.

Transforming to the Fourier representation with respect to the transverse coordinates in Eqs. (2.2) and (2.3),

$$\zeta(\mathbf{r}, t) = \int d\mathbf{k}\zeta_{\mathbf{k}}(t)e^{i\mathbf{k}r}, \qquad \psi(\mathbf{r}, t) = \int d\mathbf{k}\psi_{\mathbf{k}}(t)e^{i\mathbf{k}r} \tag{2.6}$$

and introducing the complex amplitudes a_k of the normal oscillations

$$\zeta_{\mathbf{k}} = \frac{1}{2\pi}\sqrt{\frac{\tilde k}{2\omega_k}}(a_{\mathbf{k}} + a^*_{-\mathbf{k}}), \qquad \psi_{\mathbf{k}} = -\frac{i}{2\pi}\sqrt{\frac{\omega_k}{2\tilde k}}(a_{\mathbf{k}} - a^*_{-\mathbf{k}}) \tag{2.7}$$

$$\tilde k \equiv k\,\mathrm{th}\,kh$$

$$\omega_k = \left[\left(g + \frac{\alpha}{\rho}k^2\right)\tilde k\right]^{1/2} \tag{2.8}$$

where ω_k is the dispersion law for the surface waves, we write the energy $E \equiv \rho H\{a\}$ of the roughness in the form

$$\mathcal{H} = \int d\mathbf{k}\omega_k a_{\mathbf{k}}^* a_{\mathbf{k}} + \frac{1}{3}\int d1d2d3V^{\sigma_1\sigma_2\sigma_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a^{\sigma_1}_{\mathbf{k}_1}a^{\sigma_2}_{\mathbf{k}_2}a^{\sigma_3}_{\mathbf{k}_3}\delta\left(\sum_{i=1}^{3}\sigma_i k_i\right) +$$
$$+ \frac{1}{4}\int d1d2d3d4V^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a^{\sigma_1}_{\mathbf{k}_1}a^{\sigma_2}_{\mathbf{k}_2}a^{\sigma_3}_{\mathbf{k}_3}a^{\sigma_4}_{\mathbf{k}_4}\delta\left(\sum_{i=1}^{4}\sigma_i k_i\right) \tag{2.9}$$
$$\int d1 \equiv \sum_{\sigma=\pm}\int d\mathbf{k}_1 \ \text{etc.}$$

Here we have used the notation

$$a_{\mathbf{k}}^{\sigma} = \begin{cases} a_{\mathbf{k}}^{-} \equiv a_{\mathbf{k}} & \text{for} \quad \sigma = - \\ a_{\mathbf{k}}^{+} \equiv a_{\mathbf{k}}^{*} & \text{for} \quad \sigma = + \end{cases} \tag{2.10}$$

In the transformation from Eq. (2.5) to (2.9), we used the relation between ψ_k and $\varphi_k \equiv (2\pi)^{-2}\int d\mathbf{r}\varphi(\mathbf{r}, 0; t)\exp(-i\mathbf{k}\mathbf{r})$,

$$\varphi_{\mathbf{k}} = \psi_{\mathbf{k}} - \int d\mathbf{k}_1 d\mathbf{k}_2\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\tilde k_1\psi_{\mathbf{k}_1}\zeta_{\mathbf{k}_2} +$$
$$+ \frac{1}{2}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)[\tilde k_1|\widetilde{\mathbf{k} - \mathbf{k}_2}| + \tag{2.11}$$
$$+ \tilde k_1|\widetilde{\mathbf{k} - \mathbf{k}_3}| - k_1^2]\,\psi_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}$$

correct to terms of the order of $\sim a_k^3$.

In accordance with Eq. (2.4), the equation of motion in the variables a_k^σ reduce to

$$\dot a_{\mathbf{k}}^{\sigma} = i\sigma\frac{\delta\mathcal{H}}{\delta a_k^{-\sigma}} \quad \text{or} \quad \dot a_{\mathbf{k}}^{\sigma} = \{\mathcal{H}, a_{\mathbf{k}}^{\sigma}\} \tag{2.12}$$

The variables a_k^σ are classical analogs of the creation and annihilation operators a_k^+ and a_k^- for waves in states with the wave vector **k**. See the Poisson brackets { } for the quantities a_k^σ in Eq. (3.5).

As is clear from Eq. (2.9), the matrix elements are symmetric with respect to permutation of the arguments (σ, $\mathbf{k}$), they are real because of the choice of phase in Eq. (2.7), and, therefore, do not change when there is a change in sign of all upper indices. Because of the isotropy of the medium, they are invariant with respect to simultaneous rotation of all wave vectors and, in particular, for a change in the sign of $\{k\}$. The explicit expressions for the matrix elements have the form

$$V^{\sigma_1\sigma_2\sigma_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = \frac{1}{2\pi}\frac{1}{8\sqrt{2}}\left(\frac{\omega_1\omega_2\omega_3}{\tilde{k}_1\tilde{k}_2\tilde{k}_3}\right)^{1/2}\sum\hat{P}\,\frac{\tilde{k}_1}{\omega_1}\,[\mathbf{k}_2\mathbf{k}_3+\sigma_2\sigma_3\tilde{k}_2\tilde{k}_3] \tag{2.13}$$

$$V^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = \frac{1}{(2\pi)^2}\frac{1}{16}\left(\frac{\tilde{k}_1\tilde{k}_2\tilde{k}_3\tilde{k}_4}{\omega_1\omega_2\omega_3\omega_4}\right)^{1/2}\frac{1}{4!}\sum\hat{P}\times$$
$$\times\left[\frac{\omega_3\omega_4\sigma_3\sigma_4}{\tilde{k}_3\tilde{k}_4}X(-\sigma_1\mathbf{k}_1\,|\,\sigma_3\mathbf{k}_3,\sigma_4\mathbf{k}_4)-\sigma_1\sigma_2\sigma_3\sigma_4\frac{\alpha}{2\rho}(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}_3\mathbf{k}_4)\right] \tag{2.14}$$

where $\Sigma\hat{P}$ is the sum over all permutations. The function X appearing in Eq. (2.14) is given by

$$X(\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2) = k_1^2\tilde{k}_2 + k_2^2\tilde{k}_1 - \tilde{k}_1\tilde{k}_2\,(|\widetilde{\mathbf{k}-\mathbf{k}_2}|+|\widetilde{\mathbf{k}-\mathbf{k}_1}|) \tag{2.15}$$

and has the following symmetry properties:

$$X(\mathbf{k}\,|\,\mathbf{k}_1,\ \mathbf{k}_2) = X(\mathbf{k}\,|\,\mathbf{k}_2,\ \mathbf{k}_1) = X(-\mathbf{k}\,|\,-\mathbf{k}_1,\ -\mathbf{k}_2) \tag{2.16}$$

The matrix element (2.14) is conveniently rewritten (considering $\Sigma\sigma_i\mathbf{k}_i = 0$), having replaced $X(-\sigma_1\mathbf{k}_1|\sigma_3\mathbf{k}_3,\ \sigma_4\mathbf{k}_4)$ by a function which is symmetric with respect to each pair of arguments,

$$Y(\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2\,|\,\sigma_3\mathbf{k}_3,\sigma_4\mathbf{k}_4) = \frac{1}{2}\{2k_3^2\tilde{k}_4+2k_4^2\tilde{k}_3-$$
$$-\tilde{k}_3\tilde{k}_4\,(|\widetilde{\sigma_1\mathbf{k}_1+\sigma_3\mathbf{k}_3}|+|\widetilde{\sigma_1\mathbf{k}_1+\sigma_4\mathbf{k}_4}|+|\widetilde{\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3}|+|\widetilde{\sigma_2\mathbf{k}_2+\sigma_4\mathbf{k}_4}|)\} \tag{2.17}$$

In deep water ($kh \gg 1$), $\tilde{k}$ transforms into k in Eqs. (2.13)-(2.15). This case has been discussed [1, 2, 5].

3. The dispersion law (2.8) for capillary-gravitational waves allows the decay process

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \tag{3.1}$$

for sufficiently large $k \geq \bar{k} \sim k_0$ (if $k_0h \gg 1$, $\bar{k} = \sqrt{2}k_0$). Scattering processes such as (3.1) with the participation of one gravitational and two capillary waves are also allowed. It is impossible to satisfy Eq. (3.1) if $k < \bar{k}$. Furthermore, scattering processes are possible with conservation of the number of waves

$$\omega(\mathbf{k}) + \omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3),\ \mathbf{k}+\mathbf{k}_1 = \mathbf{k}_2+\mathbf{k}_3 \tag{3.2}$$

which are important in the gravitational region. In the consideration of nondecay processes, it is convenient to make a canonical transformation which eliminates from the Hamiltonian $\mathscr{H}$(2.9) cubic terms that do not make a contribution to the transition probability in first-order perturbation theory. In the present case, such a transformation cannot be made in the entire $\mathbf{k}$ space because of the appearance of divergences associated with the decay nature of the spectrum when $k > \bar{k}$. We therefore make a transformation to new variables symbolically written in the form [see Eq. (3.9) also]

$$A_{\mathbf{k}}^{\sigma} = e^{-S}a_{\mathbf{k}}^{\sigma}e^{S} \tag{3.3}$$

so as to eliminate from the Hamiltonian only forbidden triple processes. In that case, A_k^{σ} will differ from a_k^{σ} only when $k < \bar{k}$. Denoting the quadratic, cubic, and fourth-power terms with respect to the variable a_k^{σ} in the Hamiltonian (2.9), respectively, by $\mathscr{H}_i(a)$ ($i = 2, 3, 4$), we separate out in $\mathscr{H}_3$ the term $\overline{\mathscr{H}}_3$ responsible for forbidden processes. In the new variables, the Hamiltonian $\tilde{\mathscr{H}}(A) \equiv \mathscr{H}(a) = e^{S(A)}\mathscr{H}(A)e^{-S(A)}$ should not contain $\overline{\mathscr{H}}_3$. Expanding e^S in a series in terms of small S, we obtain

$$\tilde{\mathscr{H}} = \mathscr{H}_2+(\mathscr{H}_3+[S,\mathscr{H}_2])+\left(\frac{1}{2}[S[S,\mathscr{H}_2]]+[S,\mathscr{H}_3]+\mathscr{H}_4\right)+O(A^5) \tag{3.4}$$

where the square brackets denote a Poisson bracket divided by i and are calculated using relations invariant with respect to canonical transformations:

$$\frac{1}{i}\{a_{\mathbf{k}}^{\sigma}, a_{\mathbf{k}'}^{\sigma'}\} \equiv [a_{\mathbf{k}}^{\sigma}, a_{\mathbf{k}'}^{\sigma'}] = \sigma'\delta_{\sigma,-\sigma'}\delta(\mathbf{k}-\mathbf{k}') \tag{3.5}$$

As is clear from Eq. (3.4), the term $\overline{\mathscr{H}}_3$ vanishes if S is defined by the equality

$$\overline{\mathscr{H}}_3 + [S,\ \mathscr{H}_2] = 0 \tag{3.6}$$

which leads to

$$S=\frac{1}{3}\int d1d2d3S_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}a_{\mathbf{k}_1}^{\sigma_1}a_{\mathbf{k}_2}^{\sigma_2}a_{\mathbf{k}_3}^{\sigma_3}\delta\left(\sum_{i=1}^{3}\sigma_i\mathbf{k}_i\right) \tag{3.7}$$

$$S_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}=\left(\sum_{i=1}^{3}\sigma_i\omega_i\right)^{-1}\tilde{V}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}$$

The transformation (3.3) is canonical, since $S_k^{\sigma} = -S_k^{\sigma}$ in view of Eqs. (3.7) and (2.13) (i.e., S is anti-Hermitian, and e^S is a unitary matrix). We set the diagonal part of S, which is not defined by Eq. (3.6), equal to zero. In the derivation of the canonical transformation, it is convenient to use a quantum analogy. We compare to this classical system a Bose gas with the Hamiltonian (2.9), where the a_k^{σ} are the creation operator a_k^+ and the annihilation operator a_k^- obeying the commutation rules (3.5). The unitary transformation (3.3), where S is an anti-Hermitian matrix, corresponds to the classical canonical transformation. Expanding e^S in a series in terms of the operators S, one can arrive at Eq. (3.4), which corresponds to the classical form if the commutator is replaced by the Poisson bracket (3.5).

Thus, we arrive at the effective Hamiltonian

$$\tilde{\mathscr{H}}=\int d\mathbf{k}\omega_{\mathbf{k}}A_{\mathbf{k}}^{*}A_{\mathbf{k}}+\frac{1}{3}\int d1d2d3\bar{V}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}A_{\mathbf{k}_1}^{\sigma_1}A_{\mathbf{k}_2}^{\sigma_2}A_{\mathbf{k}_3}^{\sigma_3}\delta\left(\sum_{i=1}^{3}\sigma_i\mathbf{k}_i\right)+ \tag{3.8}$$
$$+\frac{1}{4}\int d1d2d3d4\tilde{V}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^{\sigma_1\sigma_2\sigma_3\sigma_4}A_{\mathbf{k}_1}^{\sigma_1}A_{\mathbf{k}_2}^{\sigma_2}A_{\mathbf{k}_3}^{\sigma_3}A_{\mathbf{k}_4}^{\sigma_4}\delta\left(\sum_{i=1}^{4}\sigma_i\mathbf{k}_i\right)$$

where the new normal coordinates are associated with the old ones by the relations

$$A_{\mathbf{k}}^{\sigma}=a_{\mathbf{k}}^{\sigma}-[S,a_{\mathbf{k}}^{\sigma}] \tag{3.9}$$
$$A_{\mathbf{k}}^{-\sigma}=a_{\mathbf{k}}^{-\sigma}+\sigma\int d1d2\delta\left(\sum_{i=0}^{2}\sigma_i\mathbf{k}_i\right)\tilde{V}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\sigma\sigma_1\sigma_2}\left(\sum_{i=0}^{2}\sigma_i\omega_i\right)^{-1}a_{\mathbf{k}_1}^{\sigma_1}a_{\mathbf{k}_2}^{\sigma_2}\qquad(\sigma\equiv\sigma_0)$$

The effective matrix element is

$$\tilde{V}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^{\sigma_1\sigma_2\sigma_3\sigma_4}=V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^{\sigma_1\sigma_2\sigma_3\sigma_4}+\frac{2}{4!}\sum\hat{P}\int d5\sigma_5\tilde{V}_{\mathbf{k}_5\mathbf{k}_3\mathbf{k}_4}^{\sigma_5\sigma_3\sigma_4}\tilde{V}_{\mathbf{k}_5\mathbf{k}_1\mathbf{k}_2}^{-\sigma_5\sigma_1\sigma_2}\frac{\delta(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2-\sigma_5\mathbf{k}_5)}{\sigma_1\omega_1+\sigma_2\omega_2-\sigma_5\omega_5} \tag{3.10}$$

In this case, the matrix element $\bar{V}$ corresponds to forbidden processes

$$\bar{V}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}=\begin{cases}V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\sigma_1\sigma_2\sigma_3}, & \text{if}\quad k_{1,2,3}<\bar{k}\\ 0 & \text{otherwise}\end{cases} \tag{3.11}$$

and $\tilde{V} \equiv V - \bar{V}$ corresponds to allowed triple processes. Terms are omitted in Eq. (3.10), which contain the product $\tilde{V}\bar{V}$, that describe the interaction of capillary and gravitational waves to higher order than the terms of third order $\sim\tilde{V}$ in the Hamiltonian (3.8).

The kinetic equation for the number of waves [4] (quasiparticles) $N(\mathbf{k})$

$$N_k=I^{(3)}\{N\}+I^{(4)}\{N\} \tag{3.12}$$

arises in the random-phase approximation

$$\langle A_{\mathbf{k}}^{\sigma}A_{\mathbf{k}'}^{\sigma'}\rangle=N(\mathbf{k})\,\delta_{\sigma,-\sigma'}\delta(\mathbf{k}-\mathbf{k}') \tag{3.13}$$

where < > denotes an average over an ensemble, and the collision integrals $I\{N\}$ describe the change in the number of waves because of nonlinear interactions.

It is also convenient to use a quantum analogy [3] in the derivation of Eq. (3.12), although it can be obtained by other means [4]. Using the relations between the canonical variable A_k^{σ} and the annihilation and creation operators, the collision term can be written down directly, avoiding laborious calculation. The collision integral describes the balance between quasiparticle entrance into, and departure from, the state **k** and is expressed through the probabilities for the corresponding processes. The transition probability is 2π multiplied by the square of the modulus of the matrix element of the Hamiltonian (3.8) (we assume $\hbar = 1$), and the N combination appearing in f can be discovered by comparing f with its quantum analog, which for decay and the inverse confluence (3.1) has the form

$$f^{q}(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2)=(N+1)N_1N_2-(N_1+1)(N_2+1)N$$

and for the scattering processes (3.2) is

$$f^q(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3) = (N+1)(N_1+1)N_2N_3 - NN_1(N_2+1)(N_3+1)$$

The factor N + 1 corresponds to the creation of a quasiparticle (wave) and the factor N, to its annihilation in a given process. These elements arise from the matrix elements of the operators A_k^{σ}. The combinations f^q are selected because of the equal probabilities for the direct and inverse processes (principle of detailed balance) which corresponds to $V_k^{\sigma} = V_k^{-\sigma}$ in Eqs. (2.9) and (3.8). The conversion from f^q to f corresponds to the condition $N \gg 1$ and a return to the normalization (2.9).

The term

$$I^{(3)}\{N\} = \int d\mathbf{k}_1 d\mathbf{k}_2 [W_{\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2} f(\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2) - W_{\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}} f(\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}) - W_{\mathbf{k}_2\,|\,\mathbf{k}\mathbf{k}_1} f(\mathbf{k}_2\,|\,\mathbf{k}\mathbf{k}_1)] \tag{3.14}$$

describes the triple (decay) processes

$$W_{\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2} = \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega-\omega_1-\omega_2)\,U_{\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2}$$
$$U_{\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2} = 4\pi\,|V^{+--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2 \tag{3.15}$$

the transition probability, and

$$f(\mathbf{k}/\mathbf{k}_1\mathbf{k}_2) = N_1N_2 - NN_1 - NN_2,\quad N_1 \equiv N(\mathbf{k}_1)\ \text{etc.} \tag{3.16}$$

The collision integral for quadruple processes

$$I^{(4)}\{N\} = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 W_{\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3} f(\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3) \tag{3.17}$$

describes the scattering of a gravitational wave,

$$W_{\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3} = \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,U_{\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3} \tag{3.18}$$
$$U_{\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3} = \frac{2\pi}{2}\,|3!\tilde{V}^{++--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2$$
$$f(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3) = N_1N_2N_3 + NN_2N_3 - NN_1N_2 - NN_1N_3 \tag{3.19}$$

The triple-process collision integral (3.14) agrees with that obtained in [2] for waves in deep water ($kh \gg 1$) when $k \gg k_0$, and the quadruple-process integral (3.17) agrees with the collision term in [1]. The notation in [6-8] was used in writing the collision integrals (3.14)-(3.19) for convenience in comparison. We give here expressions and estimates for matrix elements needed in the following:

$$V^{\sigma_1\sigma_2\sigma_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = \frac{1}{2\pi}\frac{1}{8\sqrt{2}}\left(\frac{\omega_1\omega_2\omega_3}{\tilde{k}_1\tilde{k}_2\tilde{k}_3}\right)^{1/2}\sum \hat{P}\,\frac{\tilde{k}_1}{\omega_1}(\mathbf{k}_2\mathbf{k}_3),\ kh \ll 1 \tag{3.20}$$

$$V^{+--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{1}{8\pi\sqrt{2}}\left(\frac{\alpha}{\rho}\right)^{1/4}(kk_1k_2)^{1/4}\left[\frac{1}{\sqrt{k}}(\mathbf{k}_1\mathbf{k}_2 + k_1k_2) + \right. \tag{3.21}$$
$$\left. + \frac{1}{\sqrt{k_1}}(\mathbf{k}\mathbf{k}_2 - kk_2) + \frac{1}{\sqrt{k_2}}(\mathbf{k}\mathbf{k}_1 - kk_1)\right] \sim k^{9/4},\ k \gg k_0, h^{-1}$$
$$V^{(3)}_k \sim \omega_k^{1/2}k^{3/2},\quad k \gg h^{-1},\quad k \lesssim k_0$$
$$V^{+--}_{\mathbf{k}\mathbf{k}_1q} = \frac{1}{4\pi\sqrt{2}}\,g^{1/4}q^{-1/4}[\mathbf{k}\mathbf{q} + O(q^2)],\quad k \gg k_0 \gg q \gg h^{-1} \tag{3.22}$$

$$V^{+--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{1}{8\pi\sqrt{2}}\left(\frac{\alpha}{ph}\right)^{1/4}(k^2 + \mathbf{k}_1\mathbf{k}_2),\quad h^{-1} \gg k \gg k_0 \tag{3.23}$$
$$V^{++--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim \tilde{V}^{++--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim k^3,\quad k_0 \gg k \gg h^{-1} \tag{3.24}$$
$$V^{++--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim h^{-1}k^2,\quad k \ll k_0, h^{-1} \tag{3.25}$$
$$V^{+--}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \sim k_1^{7/4}k^{1/2},\quad k, k_2 \gg k_1 \gg k_0, h^{-1} \tag{3.26}$$
$$U_{\mathbf{k}\mathbf{k}_1\,|\,\mathbf{k}_2\mathbf{k}_3} \sim (k_1k_3)^2\,kk_2,\quad k_0 \gg k, k_2 \gg k_1, k_3 \gg h^{-1} \tag{3.27}$$

The matrix elements given contain complete information about nonlinear interaction of surface waves. They determine the growth rate of decay instabilities [5], the amplitudes of the highest harmonics, etc. The asymptotes (3.26) and (3.27) are needed for studies of the convergence of the collision integrals (see [7, 8]). The law for conservation of momentum was used in the expressions [see Eqs. (2.9), (3.8)].

4. For a thin layer of fluid with a depth h less than k_0^{-1}, we determine the parameters of weak turbulence in the wavelength region where $kh \ll 1$ (shallow water). The wave-dispersion law

$$\omega(k)=\sqrt{\frac{\alpha}{\rho}hk^2}, \quad h^{-1}\gg k\gg k_0 \tag{4.1}$$

and the transition probability W_k, which according to Eqs. (3.15) and (3.20) is

$$W_{\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2}=\frac{1}{32\pi}\sqrt{\frac{\alpha}{\rho h}}k^4\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega-\omega_1-\omega_2) \qquad h^{-1}\gg k\gg k_0 \tag{4.2}$$

are homogeneous functions in this region. As has been shown [6, 7], this makes it possible to find the turbulence distribution.

The stationary distribution is found as the solution of the nonlinear integral equation $I^{(3)}\{N\}=0$ which, in addition to the equilibrium solution $N\sim\omega^{-1}$, has a power solution corresponding to a constant flow of energy P over the turbulence spectrum. The exponent is determined by the degree of homogeneity of the transition probability, the dispersion law, and the dimensionality of **k** space. The solution has the form [7]

$$N\sim\omega^s, \quad s=-\frac{1}{2\beta}(m+2d) \tag{4.3}$$

Here β and m are indices of the homogeneity of the dispersion law $\omega_k\sim k^\beta$ and of the square of the modulus of the matrix element $U_k\equiv U_{k/k_1k_2}$(3.15) ($U_{\lambda k}=\lambda^m U_k$). For capillary waves in shallow water, $\beta=2$, $m=4$ and we find $s=-2$ from (4.2). The dependence of the distribution on energy flow is determined by the kinetic equation

$$\partial P/\partial k\equiv-\omega k I^{(3)}\{N\},\ I^{(3)}\{N\}\sim N^2$$

Having set up the dependence on the depth h from consideration of dimensionality, we obtain (to the order of a numerical factor ~1)

$$N=P^{1/2}\left(\frac{k}{\omega}\right)^{1/2}k^{-4}(kh)^{1/2}\sim P^{1/2}k^{-4}, \quad h^{-1}\gg k\gg k_0 \tag{4.4}$$

The solution (4.4) can be obtained on the basis of the estimate $I^{(3)}\{N\}\sim k^2\omega^{-1}N^2(k)\times U_h^{(3)}$ using the localizability of the distribution, from which it follows $N\sim P^{1/2}(k^4U_k^{(3)})^{-1/2}$ in agreement with (4.4). The distribution of capillary waves can encompass both deep and shallow water regions. Then, in accordance with Eqs. (1.2) and (4.4)

$$N=P^{1/2}\left(\frac{k}{\omega}\right)^{1/2}k^{-4}\varphi(kh), \quad \varphi(x)=\begin{cases}\sqrt{x}, & x\ll 1\\ 1, & x\gg 1\end{cases}, \quad k\gg k_0 \tag{4.5}$$

The distribution (4.4) is a local distribution, which can be confirmed by evaluating the convergence of the collision integral as was done in [7] for $\beta<2$. We note that the system of capillary waves in shallow water is a two-dimensional gas of quasiparticles with a quadratic dispersion law [$\varepsilon=p^2/2M$, $p=\hbar k$, $M=(h/2)[\alpha/\rho h]^{-1/2}$, ħ is Planck's constant]. Decays occur at a right angle, and the transition probability (4.2) has a simple form. Capillary waves in shallow water can serve as a convenient model for the study of stochastic systems.

5. Coupling of turbulence spectra when $k\sim k_0$ is of fundamental interest. We consider the case $k_0h\gg 1$ and investigate the solution with constant flow of energy passing from the gravitational region into the capillary region. The transition into the region of decay spectra is accompanied by a change in the dependence of N on the flow P. Therefore, the distribution is expressed through a function of the two dimensionless parameters k/k_0 and P/V_k^3 ($V_k\equiv\omega/k$), the asymptotic form of which is obtained from comparison with Eqs. (1.1) and (1.2),

$$N=P^{1/3}k^{-4}F\left(\frac{P}{V_k^3},\frac{k}{k_0}\right), \quad F(x,y)=\begin{cases}1, & y\ll 1\\ x^{1/6}, & y\gg 1\end{cases} \tag{5.1}$$

The parameter P/V_k^3 appearing here plays an important role in the theory of weak turbulence; weakness of turbulence (smallness of flow) corresponds to small values of this dimensionless parameter, $P/V_k^3\ll 1$. We turn to the effects resulting from the interaction of capillary and gravitational waves. The scattering of capillary waves by gravitational waves, which is contained in $I^{(3)}$, can be significant in the region adjacent to

$k_0 (k \leq k_0)$. These processes lead to a change in the spectrum of gravitational waves [9] and to their damping [10]. Denoting the wave vector for a gravitational wave by **q** and that for a capillary wave by **k**, we write the conservation law corresponding to this process:

$$\omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{q}), \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{q} \tag{5.2}$$

Confining ourselves to the deep-water case, $kh \gg 1$ and $qh \gg 1$, we assume that a weak-turbulence distribution is established in the capillary region. For small flows of energy, the quadruple processes of gravitational-wave scattering can be neglected [$I^{(4)} \ll I^{(3)}$]. Therefore, the turbulence spectrum in the gravitational region will be induced by the spectrum $n(\mathbf{k})$ of capillary waves (1.2), and a stationary distribution N_q of gravitational waves is found from [9]

$$I^3\{N_\mathbf{q}, n\} = -\int d\mathbf{k}_1\, d\mathbf{k}_2\, [W_{\mathbf{k}_1/\mathbf{k}_2\mathbf{q}}\,(N_\mathbf{q} n_2 - N_\mathbf{q} n_1 - n_1 n_2) + W_{\mathbf{k}_2|\mathbf{k}_1\mathbf{q}}\,(N_\mathbf{q} n_1 - N_\mathbf{q} n_2 - n_1 n_2)] = 0 \tag{5.3}$$

The integration in Eq. (5.3) is carried out over the region $k_{1,2} \geq \sqrt{2k_0}$ in accordance with Eq. (3.15), $n_1 \equiv n(\mathbf{k}_1)$. The term corresponding to the decay of gravitational waves into capillary waves and proportional to $W_{q|k_1k_2}$ is absent, since this process is forbidden by the conservation laws. The second term in Eq. (5.3), which is proportional to $W_{k_2|k_1q} f(\mathbf{k}_2|\mathbf{k}_1\mathbf{q})$, reduces to the first term through the substitution of variables $\mathbf{k}_1 \rightleftarrows \mathbf{k}_2$ so that we have for the induced distribution of gravitational waves

$$N_\mathbf{q} = \left[\int d\mathbf{k}_1\, d\mathbf{k}_2 n_1 n_2 W_{\mathbf{k}_1|\mathbf{k}_2\mathbf{q}}\right]\left[\int d\mathbf{k}_1\, d\mathbf{k}_2\,(n_2 - n_1)\, W_{\mathbf{k}_1|\mathbf{k}_2\mathbf{q}}\right]^{-1} \tag{5.4}$$

The distribution N_q takes on a power-law form in the region $q \ll k_0$. From Eq. (5.2) we have $k_2 \approx k_1 = 4k_0^2/9q \cos \mathbf{qk}$, $n_2 - n_1 \approx -\omega_q(\partial n_1/\partial\omega_1)$, and Eq. (5.4) reduces to

$$N_q \approx P^{1/2}\left(\frac{k_0}{\omega_0}\right)^{1/2} k_0^{-4}\left(\frac{q}{k_0}\right)^{9/4}, \quad \omega_0 \equiv \omega(k_0) \tag{5.5}$$

Thus, a falling spectrum in the capillary region, $n_k \sim k^{-17/4}$, induces a rising spectrum in the gravitational region, $N_q \sim q^{9/4}$ [9].

The distribution (5.5) is only possible for small energy flows $P/V_0^3 \ll 1$, since it is impossible to consider the contribution small from processes of higher order in the neighborhood of k_0 when $P/V_0^3 > 1$ and the turbulence is not weak. For scattering of capillary waves by gravitational waves, the kinetic equation $\dot{N}_q = I^{(3)}$ is a linear differential equation because of Eq. (5.3). The nonstationary problem of the interaction of gravitational waves with an arbitrary ensemble of capillary waves is solved exactly; the inverse relaxation time is equal to twice the denominator in Eq. (5.4) (compare [10]).

6. As follows from the kinetic equation, weak turbulence corresponds to small values of the parameter $k^2\omega_k^{-1}U_k^{(3)}N_k$ or $k^4\omega_k^{-1}U_k^{(4)}N_k^2$, when the collision frequency is less than the wave frequency. When there is an increase in this parameter, interaction processes of higher order, which are not taken into account in the kinetic equation, begin to play an important role. For a distribution with constant flows, this denotes the smallness of P/V_k^3 or $\omega Q/V_k^3$. The condition of weak turbulence begins to break down (with increase in flow) apparently locally and primarily in the neighborhood of $k \sim k_0$, since the phase velocity is minimal when $k = k_0$. This is also seen from a comparison of weak-turbulence distributions with the universal distributions of Phillips and Hicks [11]. The distribution [11] leading to the wave-number spectrum $\Psi(k) = (B/\pi)k^{-4}$ was obtained from dimensionality considerations and independently of considerations associated with the stability of a water surface. The corresponding $N(\mathbf{k})$ is

$$N(\mathbf{k}) = \frac{B}{\pi} V(k) k^{-4}, \quad V(k) \equiv \omega / k, \quad B \sim 10^{-3}$$
$$\langle \zeta(\mathbf{r}, t)\, \zeta(\mathbf{r} + \xi, t)\rangle = \int d\mathbf{k}\,\Psi(\mathbf{k})\, e^{i\mathbf{k}\xi}, \quad \Psi(\mathbf{k}) = N(\mathbf{k}) / (2\pi)^2 V(\mathbf{k}) \tag{6.1}$$

The universal distribution for capillary waves has the same form with a different constant $B' \sim B$.

In contrast to the Dbukhov–Kolmogorov distribution $E_k = P^{2/3}k^{-5/3}$ for turbulence in an incompressible fluid [12], the distribution (6.1) does not contain the flow (in a system of propagating waves such distribution can be constructed from powers of ω and k which determine the local scale of turbulence). At sufficiently high flows $P/V_0^3 > 1$, the weak-turbulence distributions (1.1) and (1.2) will intersect the universal distributions. We consider the gravitational portion of the spectrum. If the source is located in the region of small wave numbers $k \sim \alpha$, and an energy flow is created toward larger k, the distribution (5.1), which transforms into the Phillips distribution for wave numbers $k_{Ph} \sim k_0(P/V_0^3)^{-2/3}$, will be valid for $k \gg \alpha$ and $P/V_k^3 < 1$. The asymptote corresponding to this (N_k does not depend on P!) is $F(P/V_k^3, k/k_0) = (P/V_k^3)^{-1/3}$ for $P/V_k^3 \gg 1$, $k/k_0 \ll 1$, which also leads to the Phillips distribution according to Eq. (5.1).

As in the preceding, when there is a large energy flow in the capillary portion of the spectrum, the distribution (5.1) transforms into the Hicks distribution when $k \lesssim k_H \sim k_0(P/V_0^3)^{2/3}$, which corresponds to the asymptote

$$F(P / V_k^3,\ k / k_0) = (P / V_k^3)^{-1/3} \text{ for } k / k_0 \gg 1,\ P / V_k^3 \gg 1$$

Thus, one can assume that the distributions [11] are limiting for distributions with constant flow of energy over the turbulence spectrum. There is no explicit dependence on flow) the magnitude of the flow determines the region of transition from a universal spectrum to a spectrum of weak turbulence.

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